Problem 19. Let $T : X \to Y$ be a bounded operator between Banach spaces X, Y such that $\dim \ker(T) < \infty$ and $\operatorname{codim} \operatorname{ran}(T) < \infty$. Show that T is a Fredholm operator, i.e. that the condition $\operatorname{ran}(T)$ is closed in the definition of a Fredholm operator is automatically satisfied.

Problem 20. Let H be a Hilbert space and $K(H) \subseteq B(H)$ be the closed ideal of compact operators on H so then $C(H) = B(H)/K(H)$ is a Banach algebra. Hence, $[T_1] = [T_2]$ if and only if $T_1 + T_2 + K$ for some compact perturbation K. Show that the following are equivalent:

(i) $[T]$ is invertible in $C(H)$,
(ii) There exists a $S \in B(H)$ such that $I - TS \in K(H)$ and $I - ST \in K(H)$,
(iii) T is Fredholm.

Problem 21. For which of the three topologies ($\| \ \|$, SOT, WOT) is the mapping

$$\begin{aligned} B(H) &\to B(H) \\ T &\mapsto T^* \end{aligned}$$

continuous?
Hint: The answer is yes for the norm and WOT, but no for SOT.
Counterexample: Let $S : \ell^2 \to \ell^2$ be the (left) shift, then $S^n \xrightarrow{SOT} 0$, but $(S^*)^n$ is not convergent with respect to SOT.

Problem 22. Let A_n, $B_n \subseteq B(H)$ be sequences of operators. Show

(a) $A_n \xrightarrow{SOT} A$, $B_n \xrightarrow{SOT} B \implies A_n B_n \xrightarrow{SOT} AB$.
(b) $A_n \xrightarrow{WOT} A$, $B_n \xrightarrow{WOT} B \;\not\Longrightarrow\; A_n B_n \xrightarrow{WOT} AB$.

Problem 23. Show that the positive square root of a positive semi-definite operator is uniquely determined.

Problem 24. Let A, $B \in B(H)$.

(a) Show that $r(AB) = r(BA)$
(b) Let $A \geq B \geq 0$. Show that $A^{1/2} \geq B^{1/2}$, whereas $A^2 \geq B^2$ does not necessarily hold.
Hint: First it can be assumed that A is invertible. The general case then follows with a limit value argument.

Problem 25. Let $V \in B(H)$. Show that the following statements are equivalent:

(i) V is a partial isometry.
(ii) V^* is a partial isometry.
(iii) V^*V is a projection (namely a projection onto the domain of V).
(iv) VV^* is a projection (namely a projection onto $\operatorname{ran} V$).
(v) $V = VV^*V$.
(vi) $V^* = V^*VV^*$.